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Phase of Two-Body Bose-Einstein Condensates with Collision

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Abstract By using of the invariant theory, we have studied the phase of two-body Bose-Einstein condensates with collision, the dynamical and geometric phases are presented respectively. The Aharonov-Anandan phase is also obtained in the case of considering the cyclical evolution.

Keywords Bose-Einstein condensation

1 Introduction

Recently, much attention has been paid to the investigation of Bose-Einstein condensation (BEC) in dilute and ultracold gases of neutral alkali-metal atoms using a combination of laser and evaporative cooling [1–7] due to the optical properties [8–17] statistical properties [18–24], phase properties [14–17, 25–37], and tunneling effect [38–53].

As we known that the quantum invariant theory proposed by Lewis and Riesenfeld [54] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized in [55] by introducing the concept of basic invariants and used to study the geometric phases [56–58] in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations [56, 57], but also provides us with new insights in many physical phenomena. The concept of Berry's phase has developed in some different directions [59–67]. In this paper, by using of the invariant theory, we shall study the dynamical and the geometric phases of two-body Bose-Einstein condensates with collision.

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2 Model

We consider a zero-temperature two-body Bose-Einstein condensates with collision, the Hamiltonian of this system is

$$\hat{H} = \sum_{j=1,2} \omega_j(t)[\hat{a}_j^\dagger \hat{a}_j + \Omega_j(t)(\hat{a}_j^\dagger \hat{a}_j)^2] + \nu_{12}(t)\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 + \gamma(t)(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), \quad (1)$$

where $\Omega_j(t)(\hat{a}_j^\dagger \hat{a}_j)^2$ stands for two-body hard-sphere collisions [68, 69]. $\Omega_j = 2\pi\hbar a_s/mV$, V^{-1} is the effective mode volume of the trap, a_s is the s-wave scattering length of the condensate j, and m is the atomic mass. $\nu_{12}(t)\hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2$ describes the collision between the atoms of the two condensates, the last one in (1) denotes the tunneling term.

3 Geometric and Dynamical Phases of Two-Body Bose-Einstein Condensates with Collision

For self-consistent, we first illustrate the Lewis-Riesenfeld (L-R) invariant theory [54]. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{I}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial \hat{I}(t)}{\partial t} + [\hat{I}(t), \hat{H}(t)] = 0. \quad (2)$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$\hat{I}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \quad (3)$$

where $\frac{\partial \lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial|\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \quad (4)$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of (4) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{I}(t)$ only by a phase factor $\exp[i\delta_n(t)]$, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \quad (5)$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (4). Then the general solution of the Schrödinger equation (4) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \quad (6)$$

where

$$\delta_n(t) = \int_0^t dt' \left\langle \lambda_n, t' \middle| i\frac{\partial}{\partial t'} - \hat{H}(t') \middle| \lambda_n, t' \right\rangle, \quad (7)$$

and $C_n = \langle \lambda_n, 0|\psi(0)\rangle_s$.

For simplicity, we consider the case of $\Omega(t) = \Omega_1(t) = \Omega_2(t) = \frac{1}{2}\nu_{12}(t)$. Then (1) can be rewritten as

$$\hat{H} = \omega_1(t)\hat{N}_1 + \omega_2(t)\hat{N}_2 + \Omega(t)(\hat{N}_1^2 + \hat{N}_2^2 + 2\hat{N}_1\hat{N}_2) + \gamma(t)(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1), \quad (8)$$

where $\hat{N}_j = \hat{a}_j^\dagger \hat{a}_j$ $(j = 1, 2)$. It is easy to find that $\hat{I}_1(t) = \hat{N}_1^2 + \hat{N}_2^2 + 2\hat{N}_1\hat{N}_2$ is a special invariant of this system and satisfies $\hat{I}_1(t)|m\rangle_{a_1}|n\rangle_{a_2} = \lambda_{mn}|m\rangle_{a_1}|n\rangle_{a_2}$, where $\hat{N}_1|m\rangle_{a_1} = m|m\rangle_{a_1}$, $\hat{N}_2|n\rangle_{a_2} = n|n\rangle_{a_2}$, and $\lambda_{mn} = m^2 + n^2 + 2mn$.

In the following, we can restrict the space being in the sub-space of the eigenstate of the invariant $\hat{I}_1(t)$. Corresponding, $\hat{I}_1(t)$ appearing in (8) can be replaced by its eigenvalue λ_{mn}.

In order to obtain the exact solutions of (4), we can define operators $\hat{K}_+$, $\hat{K}_-$ and $\hat{K}_0$ as follows:

$$\hat{K}_+ = \hat{a}_1^\dagger \hat{a}_2, \qquad \hat{K}_- = \hat{a}_2^\dagger \hat{a}_1, \qquad \hat{K}_0 = \hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2, \tag{9}$$

which hold the commutation relations

$$[\hat{K}_0, \hat{K}_\pm] = \pm 2\hat{K}_\pm, \qquad [\hat{K}_+, \hat{K}_-] = \hat{K}_0, \tag{10}$$

it is easy to prove that operators $\hat{K}_+$, $\hat{K}_-$ and $\hat{K}_0$ together with the Hamiltonian $\hat{H}$ construct a quasi-algebra.

Then we can get the L-R invariant as follows

$$\hat{I}_2(t) = \cos\theta \hat{K}_0 - e^{-i\varphi}\sin\theta \hat{K}_+ - e^{i\varphi}\sin\theta \hat{K}_-, \tag{11}$$

it is apparent that $[\hat{I}_1(t), \hat{I}_2(t)] = 0$. Here θ and φ are determined by (2) and satisfy the relations

$$\dot{\theta} = 2\gamma(t)\sin\varphi, \tag{12}$$

$$\dot{\theta}\cos\theta\sin\varphi + \dot{\varphi}\sin\theta\cos\varphi - 2\gamma(t)\cos\theta + (\omega_2 - \omega_1)\sin\theta\cos\varphi = 0, \tag{13}$$

$$\dot{\theta}\cos\theta\cos\varphi - \dot{\varphi}\sin\theta\sin\varphi + (\omega_1 - \omega_2)\sin\theta\sin\varphi = 0, \tag{14}$$

where dot denotes the time derivative.

According to the unitary transformation method [55], we can construct the unitary transformation

$$\hat{V}(t) = \exp[\sigma\hat{K}_+ - \sigma^*\hat{K}_-], \tag{15}$$

where $\sigma = \frac{\theta}{2}e^{-i\varphi}$ and $\sigma^* = \frac{\theta}{2}e^{i\varphi}$. The invariant $\hat{I}_2(t)$ can be transformed into a new time-independent operator $\hat{I}_V$:

$$\hat{I}_V = \hat{V}^\dagger(t)\hat{I}_2(t)\hat{V}(t) = \hat{K}_0. \tag{16}$$

Correspondingly, we can get the eigenvalue equation of operator $\hat{I}_V(t)$

$$\hat{I}_V|m\rangle_{a_1}|n\rangle_{a_2} = (m-n)|m\rangle_{a_1}|n\rangle_{a_2}. \tag{17}$$

In terms of the unitary transformation $\hat{V}(t)$ and the Baker-Campbell-Hausdoff formula [70]

$$\hat{V}^\dagger(t)\frac{\partial\hat{V}(t)}{\partial t} = \frac{\partial\hat{L}}{\partial t} + \frac{1}{2!}\left[\frac{\partial\hat{L}}{\partial t}, \hat{L}\right] + \frac{1}{3!}\left[\left[\frac{\partial\hat{L}}{\partial t}, \hat{L}\right], \hat{L}\right] + \frac{1}{4!}\left[\left[\left[\frac{\partial\hat{L}}{\partial t}, \hat{L}\right], \hat{L}\right], \hat{L}\right] + \cdots, \tag{18}$$

where $\hat{V}(t) = \exp[\hat{L}(t)]$, one has

$$\begin{aligned}\hat{H}_V(t) &= \hat{V}^\dagger(t)\hat{H}(t)\hat{V}(t) - i\hat{V}^\dagger(t)\frac{\partial \hat{V}(t)}{\partial t}\\ &= \Omega(t)\lambda_{mn} + \left[\omega_1(t)\cos^2\frac{\theta}{2} + \omega_2(t)\sin^2\frac{\theta}{2} - \gamma(t)\sin\theta\cos\varphi + \frac{\dot{\varphi}}{2}(1-\cos\theta)\right]\hat{a}_1^\dagger\hat{a}_1\\ &\quad + \left[\omega_1(t)\sin^2\frac{\theta}{2} + \omega_2(t)\cos^2\frac{\theta}{2} + \gamma(t)\sin\theta\cos\varphi - \frac{\dot{\varphi}}{2}(1-\cos\theta)\right]\hat{a}_2^\dagger\hat{a}_2, \end{aligned} \tag{19}$$

where λ_{nm} is the eigenvalue of operator $\hat{I}_1(t)$. It is easy to find that $\hat{H}(t)$ differs from $\hat{I}_V$ only by a time-dependent c-number factor. Thus we can get the general solution of the time-dependent Schrödinger equation (4)

$$|\Psi(t)\rangle_s = \sum_n\sum_m C_{nm}\exp[i\delta_{nm}(t)]\hat{V}(t)|m\rangle_{a_1}|n\rangle_{a_2}, \tag{20}$$

with the coefficients $C_{nm} = \langle n, m, t=0|\Psi(0)\rangle_s$. The phase $\delta_{nm}(t) = \delta^d_{nm}(t) + \delta^g_{nm}(t)$ includes the dynamical phase

$$\begin{aligned}\delta^d_{nm}(t) &= m\int_{t_0}^t\left[\omega_1(t)\cos^2\frac{\theta}{2} + \omega_2(t)\sin^2\frac{\theta}{2}\right]dt' + n\int_{t_0}^t\left[\omega_1(t)\sin^2\frac{\theta}{2} + \omega_2(t)\cos^2\frac{\theta}{2}\right]dt'\\ &\quad + \int_{t_0}^t[\Omega\lambda_{mn} - (m-n)\gamma(t)\sin\theta\cos\varphi]dt', \end{aligned} \tag{21}$$

and the geometric phase

$$\delta^g_{nm}(t) = \int_{t_0}^t (m-n)\frac{\dot{\varphi}}{2}(1-\cos\theta)dt'. \tag{22}$$

Particularly, the geometric phase becomes in the case of considering the cyclical evolution

$$\delta^g_{nm}(t) = \frac{1}{2}\oint(m-n)(1-\cos\theta)d\varphi, \tag{23}$$

which is the known geometric Aharonov-Anandan phase.

4 Conclusions

In conclusion, we have studied the phase of two-body Bose-Einstein condensates with collision by using of the L-R invariant theory, the dynamical and geometric phases are presented respectively. Especially, the Aharonov-Anandan phase appears when we consider the condition of the cyclical evolution.

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